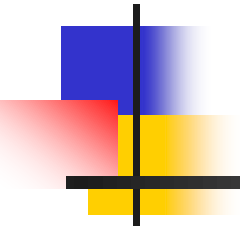


计算机科学导论

孙晓明

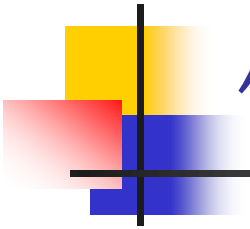
中国科学院计算技术研究所

2022-4-1

问题

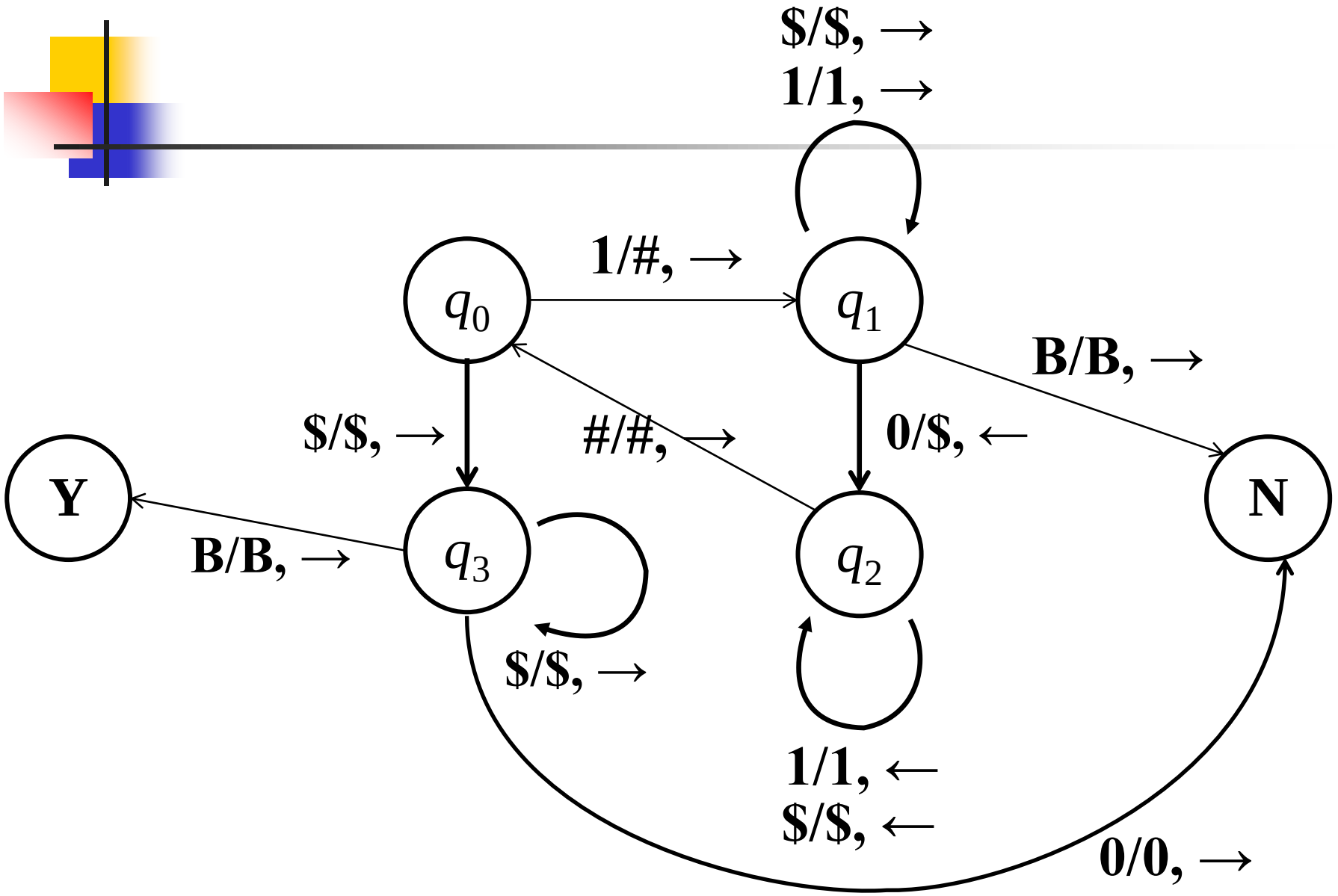
- 控制器只有一个状态、纸带上只有一个字符0的图灵机最多能运行多少步？

- 1) 2
- 2) 4
- 3) 16
- 4) $+\infty$



例4

- 输入：111…11000…00，判断1和0的个数是否相等？


$/$, →
1/1, →
1/#, →
q_0
q_1
B/B, →
$/$, →
#/#, →
0/$, ←
Y
N
B/B, →
q_3
q_2
$/$, →
1/1, ←
$/$, ←
0/0, →

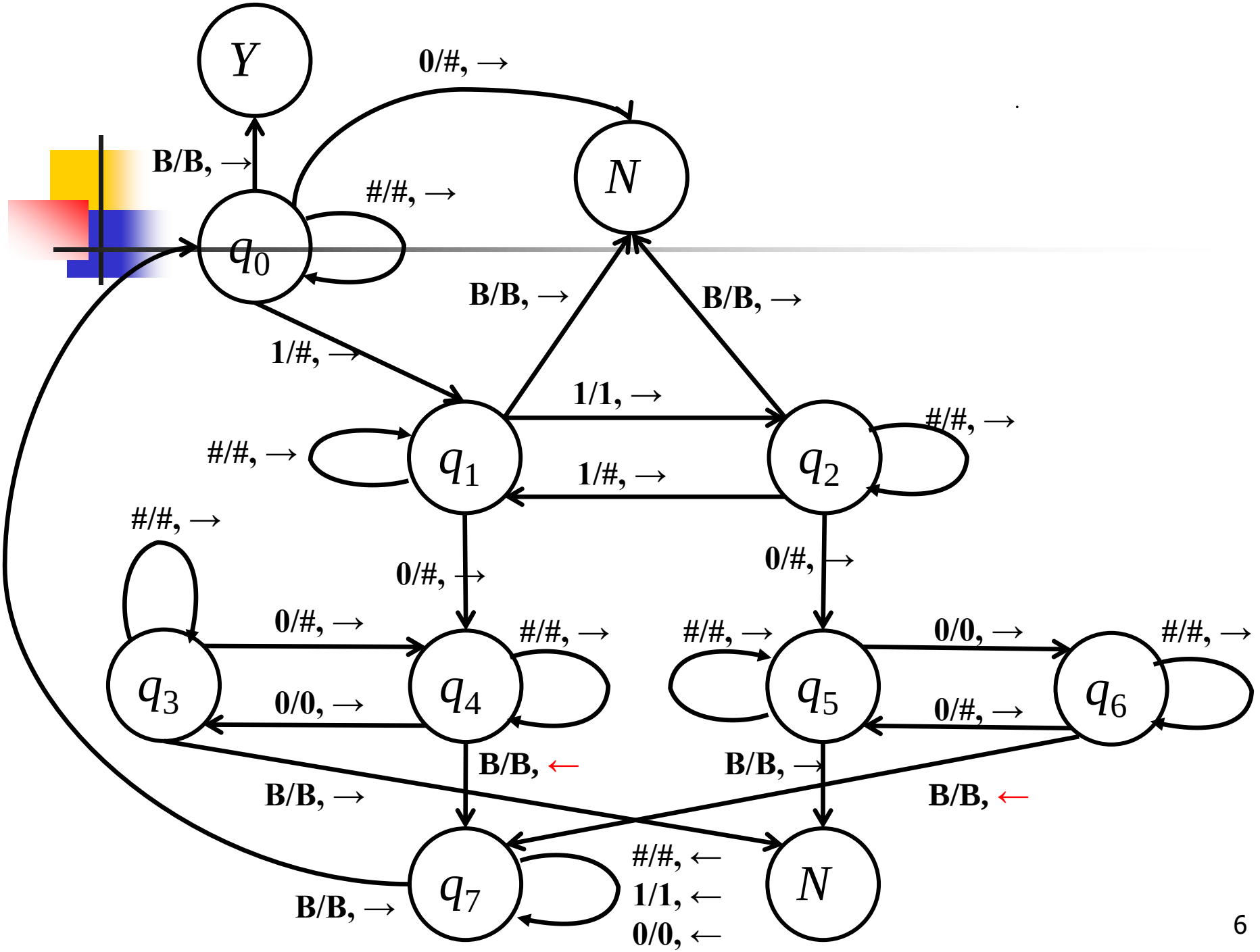
思考题

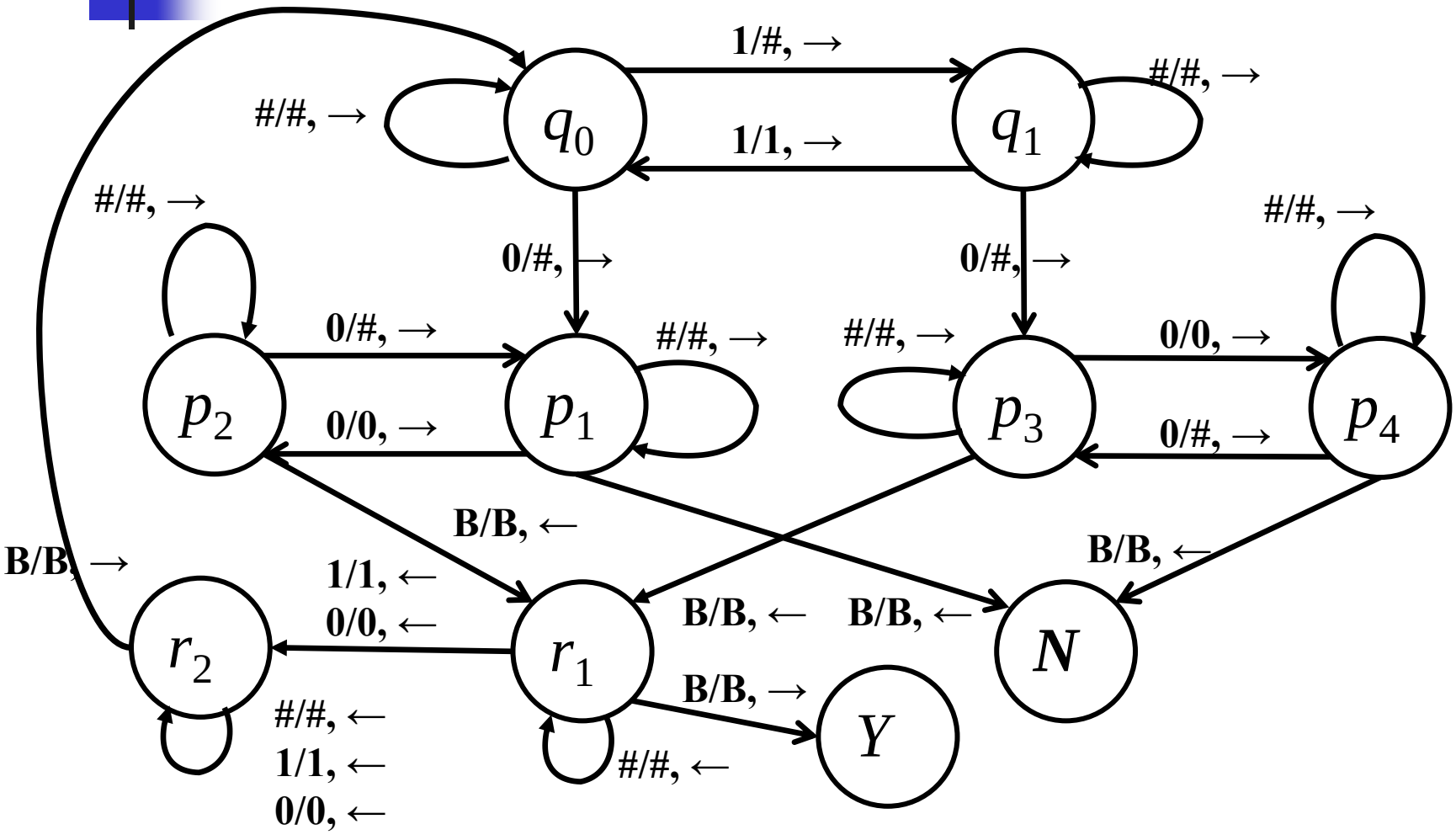
- 上面的图灵机判断 1^n0^n 需要花费~$2n^2$ 的时间，是否能够做的更快？

- Yes.
- Idea: 对于形如输入1^m0^n的输入，将m和n分别写成二进制数

  $m = m_s m_{s-1} \cdots m_1, n = n_t n_{t-1} \cdots n_1$

  然后逐位判定$m_i = n_i$，以及 $s = t$


Y
N
q_0
q_1
q_2
q_3
q_4
q_5
q_6
q_7
N
B/B, →
0/#, →
#/#, →
1/#, →
1/1, →
0/0, →
0/#, →
B/B, ←
#/#, ←
1/1, ←
0/0, ←


q_0
q_1
p_2
p_1
p_3
p_4
r_2
r_1
N
Y
1/#, →
#/#, →
1/1, →
#/#, →
#/#, →
0/#, →
0/#, →
#/#, →
0/#, →
#/#, →
0/0, →
0/0, →
0/#, →
B/B, ←
B/B, ←
B/B, →
1/1, ←
0/0, ←
B/B, ←
B/B, ←
B/B, →
#/#, ←
1/1, ←
0/0, ←
#/#, ←

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

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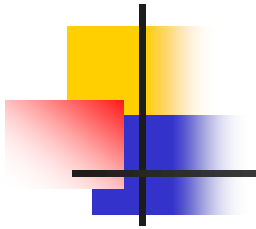
Alan Turing
1912-1954

- Turing Awards





2021 Turing Award: Jack Dongarra



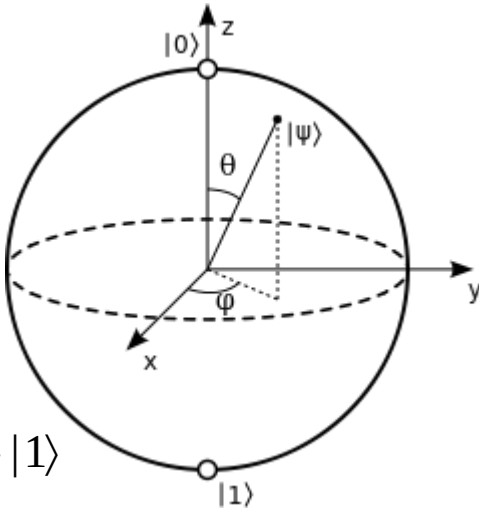
- Alonzo Church & Alan Turing:

  Church-Turing Hypothesis:

Any reasonable attempt to model mathematically computer algorithms and their performance is bound to end up with a model of computation and associated time cost that is equivalent to Turing machines within a polynomial.

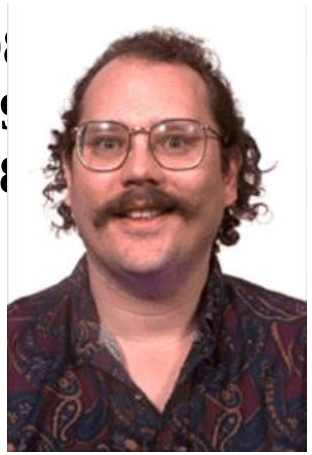
Alonzo Church
1903-1995

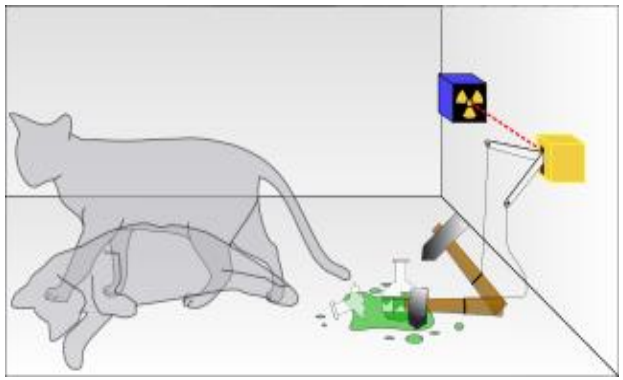
量子计算



$$\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle$$

15226050279225336053561837813263
74297180681149613806886579084945801
22963258952897654000350692006139
=37975227936943673922808
62785456553663819 9×4009
08810306837352927614683
4061





Shor's factoring algorithm (1994): $O(\log^3 N)$ $O(2^{\log^{1/3} N})$

谓词逻辑

- 存在无穷多个素数。

Euclid of Alexandria

$$\forall n, \exists m, \forall p, q\,[(m > n) \wedge (p, q > 1 \rightarrow pq \neq m)]$$

谓词逻辑

- 存在无穷多个素数

$$\forall n, \exists m, \forall p, q[(m > n$$

- 对$n > 2$，丢潘图方程 $x^n + y^n = z^n$ 不存在非平凡解。

$$\forall a, b, c, n[(abc \neq 0) \wedge (n > 2) \rightarrow a^n + b^n \neq c^n]$$

- 存在无穷多对孪生素数。

$$\forall n, \exists m, \forall p, q\,[(m > n) \wedge (p, q > 1 \rightarrow (pq \neq m) \wedge (pq \neq m + 2)]$$

- 对任何一个正整数n，如果是奇数则乘3加1，如果是偶数则除2，重复此过程，最终将得到1。

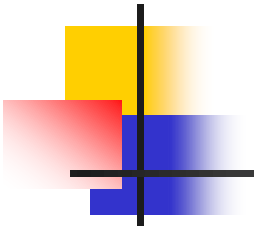
$$\forall n, \exists m, [f^{(m)}(n) = 1] \qquad f(n) = \begin{cases} 3n+1, & \text{if } n \equiv 1 (\text{mod } 2) \\ \dfrac{n}{2}, & \text{if } n \equiv 0 (\text{mod } 2) \end{cases}$$

3*n* + 1猜想

- 6 → 3 → 10 → 5 → 16 → 8 → 4 → 2 →1

- 15 → 46 → 23 → 70 → 35 → 106 → 53 →170 → 85 → 256 → 128 → 64 →32 → 16 → 8 → 4 → 2 → 1

- Solve it!!



- 四色定理：任何平面图都可以被四着色，使得任何两个相邻的顶点不同色。

$$\forall \text{平面图} G=(V,E), \exists c: V \to \{1,2,3,4\} \ \ \forall (u,v) \in E, [c(u) \neq c(v)]$$

- 图的3染色(3-coloring)：给定$G = (V, E)$

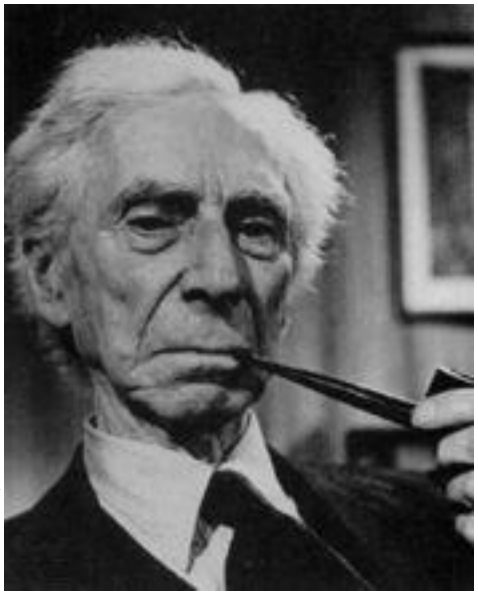
$$(\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee \neg x_4 \vee \neg x_6) \wedge (x_2 \vee x_4 \vee \neg x_8) \wedge \ldots$$

- 可满足性问题(SAT)：给定CNF公式φ

$$\exists A: \{x_1, x_2, \ldots, x_n\} \to \{0,1\}, [\varphi(x_1, \ldots, x_n) = 1]$$

理发师悖论

- 在某个城市中有一位理发师我将为本城所有不给自己理，我也只给这些人理发。”
- 问题：这位理发师是否该给



Bertrand Russell

1872—1970

罗巴切夫斯基

- 公理系统：
  - 完备性，一致性（相容性）
- 欧几里德几何：
  - 公设1：任意一点到另外任意一点可以画直线
  - 公设2：一条有限线段可以继续延长
  - 公设3：以任意点为心及任意的距离可以画圆
  - 公设4：凡直角都彼此相等
  - 公设5：同平面内一条直线和另外两条直线相交，若在某一侧的两个内角和小于二直角的和，则这二直线经无限延长后在这一侧相交。

- David Hilbert (1900)
  - Tenth Problem: determine whether a given polynomial Diophantine equation with integer coefficients has an integer solution.

$$x^{17} + y^{17} = z^{17}$$
$$x^3 + y^3 = z^3 + w^3 + u^3$$
$$x^{2020} + y^{2021} = z^{2022}$$

David Hilbert
1862-1943

Hilbert 23 problems

- 1. 连续统假设
- 2. 算术公理的相容性
- 6. 物理科学的公理化
- 8. 黎曼猜想，歌德巴赫猜想，孪生素数猜想

Entscheidungsproblem

- 公理化系统的机械化判定问题（Hilbert 1928)
  - 可判定性：是否能够找到一种有效的方法（算法），判定一个数学命题是否为真?
  - 完备性
  - 一致性

Godel不完备性定理

- 定理一：任意一个包含一阶谓词逻辑与初等数论的形式系统，都不可能同时拥有完备性和一致性。即存在一个真命题，它在这个系统中不能被证明。
- 定理二：任意一个包含初等数论的系统S，当S无矛盾时，它的无矛盾性不可能在S内证明。

Kurt Godel

1906-1978

真和可以被证明是两件事情!!

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Alan Turing
1936

可计算性

- 图灵机模型的计算能力
  - 有没有图灵机计算不了的问题？
  - Yes.

停机问题（Halting problem）：给定一台图灵机 M 和一个输入字符串 x，判定 M 在 x 这个输入上是否能够停机？（有限步终止）

没有图灵机能解决停机问题！

停机问题

- 假设有一台图灵机 M_H 能判定停机问题，即
  - $M_H(M,x)=1$，如果图灵机M在输入串x下能停机
  - $M_H(M,x)=0$，如果图灵机M在输入串x下不能停机

无论何种情况， M_H自身总是停机的！

停机问题

- 考察这样一台图灵机 $M_{\overline{H}}$(输入为x):

```
If (M_H(x, x)) {
    int i=1;
    while(true){
        i++;
    }
}
Return 1;
```

问题：在图灵机 $M_{\overline{H}}$上运行$M_{\overline{H}}$是否停机？

停机问题

111

	1	2	3	4	5	6	7	8	9	…
1	0	0	0	0	0	0	0	0	0	…
2	1	1	1	1	1	1	1	1	1	…
3	0	1	0	0	0	0	0	0	0	…
4	0	0	1	1	1	1	1	1	0	…
M_i 5	1	0	0	0	0	0	0	0	0	…
6	1	1	1	1	1	1	1	1	1	…
7	1	1	0	0	1	1	1	1	1	…
…	.	.	.	.	.	.	.	.	.	.

构造停机“真值表”：

第i号图灵机M_i

- 数字j的二进制编码串x^j
- 第i行第j列：M_i在x^j上会停机，则写1；否则写0

图灵机M_H就是在计算这张表

思考题

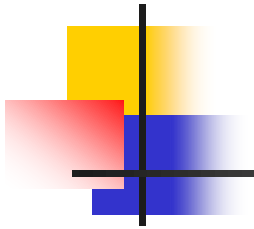
- Mr. S, Mr. P都具有足够的推理能力。约翰教授写了两个整数M和N (3≤M,N≤100)，并把M+N的值告诉了S先生，把M*N的值告诉了P先生。约翰教授问S先生和P先生：“你们能从已知的信息确定M和N的值吗？”

  S先生:” 我知道你不知道，我也不知道。”

  P先生:“现在我知道了。”

  S先生:“我也知道了。”

  请问，M和N是哪两个数？



谢谢！